An enthalpy method for moving boundary problems on the earth's surface

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Abstract

Purpose – To present a novel moving boundary problem related to the shoreline movement in a sedimentary basin and demonstrate that numerical techniques from heat transfer, in particular enthalpy methods, can be adapted to solve this problem.

Design/methodology/approach – The problem of interest involves tracking the movement (on a geological time scale) of the shoreline of a sedimentary ocean basin in response to sediment input, sediment transport (via diffusion), variable ocean base topography, and changing sea level. An analysis of this problem shows that it is a generalized Stefan melting problem; the distinctive feature, a latent heat term that can be a function of both space and time. In this light, the approach used in this work is to explore how previous analytical solutions and numerical tools developed for the classical Stefan melting problem (in particular fixed grid enthalpy methods) can be adapted to resolve the shoreline moving boundary problem.

Findings – For a particular one-dimensional case, it is shown that the shoreline problem admits a similarity solution, similar to the well-known Neumann solution of the Stefan problem. Through the definition of a compound variable (the sum of the fluvial sediment and ocean depths) a single domain-governing equation, mimicking the enthalpy formulation of a one-phase melting problem, is derived. This formulation is immediately suitable for numerical solution via an explicit time integration fixed grid enthalpy solution. This solution is verified by comparing with the analytical solution and a limiting geometric solution. Predictions for the shoreline movement in a constant depth ocean are compared with shoreline predictions from an ocean undergoing tectonic subsidence.

Research limitations/implications – The immediate limitation in the work presented here is that "off-shore" sediment transport is handled in by a "first order" approach. More sophisticated models that take a better accounting of "off shore" transport (e.g. erosion by wave motion) need to be developed.

Practical implications – There is a range of rich problems involving the evolution of the earth's surface. Many of the key transport processes are closely related to heat and mass transport. This paper

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illustrates that this similarity can be exploited to develop predictive models for earth surface processes. Such models are essential in understanding the formation of the earth's surface and could have a significant impact on natural resource (oil reserves) and land (river restoration) management.

Originality/value – For the most part the solution methods developed in this work are extensions of the standard numerical techniques used in heat transfer. The novelty of the work presented rests in the nature of the problems solved, not the method used. The particular novel feature is the time and space dependence of the latent heat function; a feature that leads to interesting analytical and numerical results.

Keywords Sedimentation, Transport management, Tectonics

Paper type Research paper

1. Introduction

A moving boundary problem is a problem in which one of the domain boundaries is an unknown. The classic example is the Stefan melting problem (Crank, 1984) a heat transfer problem requiring the tracking of the a priori unknown melting front. Since, the melt front position needs to be determined as part of the solution the problem formulation requires an additional boundary condition – the Stefan condition, obtained by balancing the net heat flux arriving at the melting front with the rate of evolution of latent heat.

Typically moving boundary problems only have a limited number of analytical solutions (Carslaw and Jaeger, 1986) and as a result, from the advent of digital computers, see Eyres *et al.* (1946) and Price and Slack (1954), there has been extensive development of numerical methods. The key feature in these methods is the mechanisms used to track the continuously moving boundary over the discrete grid of nodes that define the numerical method. Very broadly speaking, these mechanisms fall into one of three classes.

(1) *Fixed grid methods*. These methods employ a grid of nodes that remain fixed in space and track the boundary by use of an auxiliary variable. An example is the so-called enthalpy methods, used in the analysis solid-liquid phase change (Eyres *et al.*, 1946; Price and Slack, 1954; Voller and Cross, 1981; Crank, 1984; Voller *et al.*, 1990; Voller, 1996). In these methods, the melting front is tracked by the evaluation of a nodal liquid farction field, the elements of which take values $0 \leq f \leq 1$.

(2) *Deforming grid methods*. In these methods, a line of nodes is located on the moving boundary and as the solution evolves the space grid deforms to ensure that these nodes remain on the boundary (Lynch and O'Neill, 1981; Lynch, 1982; Beckett *et al.*, 2001).

(3) *Hybrid methods*. Hybrid methods employ elements of both fixed and deforming grids, e.g. local front tracking (Udaykumar *et al.*, 1999; Crank, 1957) which uses a fixed background grid and employs local front tracking schemes to follow the movement of the boundary.

Recently a new class of moving boundary problems related to tracking the evolution of sediment deposits on the earth surface has been identified. Two examples are:

(1) the evolution of sediment fans in desert environments (Paola *et al.*, 1992; Marr *et al.*, 1999, 2000; Kawakami *et al.*, 2003); and

(2) the movement of an ocean basin shoreline, on geological time scales, in response to a sediment input (Swenson *et al.*, 2000; Voller *et al.*, 2004).

This later problem can be formulated in terms of a diffusion equation and a shoreline sediment balance condition that can be interpreted as a generalization of the classic Stefan condition. Swenson *et al.* (2000), develop a deforming grid solution of this shoreline tracking problem; a solution that has been used to validate the underlying diffusion and generalized Stefan model by comparison with laboratory experiments (Paola, 2000).

In this work, it is shown that the ocean basin shoreline problem can be posed as Stefan melting problems, in which the latent heat term is a function of both space and time. This is an interesting condition, which in the strict confines of heat transfer has little if no physical basis. As a result, within the authors' knowledge, there have been no prior attempts to solve melting problems with a variable latent heat. In the context of sediment transport on the earth surface, however, the condition is physical reasonable, providing the necessary motivation and rational to develop suitable numerical solutions for variable latent heat problems. The main focus of this work is to, in the context of tracking an ocean basin shoreline, develop and present fixed grid enthalpy methods for the solution of variable latent heat Stefan melting problems. For the most part the solution methods developed are basic extensions of the standard enthalpy method (Eyres *et al.*, 1946; Price and Slack, 1954; Voller and Cross, 1981; Crank, 1984; Voller *et al.*, 1990; Voller, 1996) and the novelty of the work presented rests in the nature of the problems solved not the method used.

2. A basic shoreline problem

A basic shoreline problem involves the shoreline progradation (seaward translation) into a basin with a constant sloping basement (β) and a fixed ocean level ($z = 0$) and no tectonic subsidence of the earth's crust – the last two conditions a good approximation for modern continental margins with large sediment supply and small subsidence rates. A schematic 3D section of such a basin indicating the temporal variables is shown in Figure 1. Sediment is generated by erosion of the uplands and transported over the land surface by fluvial processes, i.e. by water flowing through a network of surface channels. In the typical time scales of problems of sedimentary geology these channels are not fixed but can migrate (braid) over the land surface and expand (flood) and contract with time. The sediment arriving at the shoreline fills the near shore region or is carried off-shore by ocean currents and wave motions. An excess of sediment arriving at the shoreline will result in an advance of the shoreline, $s(t)$; the moving boundary of interest.

Governing equations can be obtained by considering a representative 2D cross-section defined by axis of the land surface height and horizontal distance from the sediment source, see front of Figure 1. In this domain two distinct regimes can be identified in the system

(1) a sub-aerial, fluvial domain; and

(2) an offshore, submarine domain.

In the fluvial-domain, sediment transport and deposition by river systems on geological time scales is modeled by the diffusion equation (Swenson *et al.*, 2000).

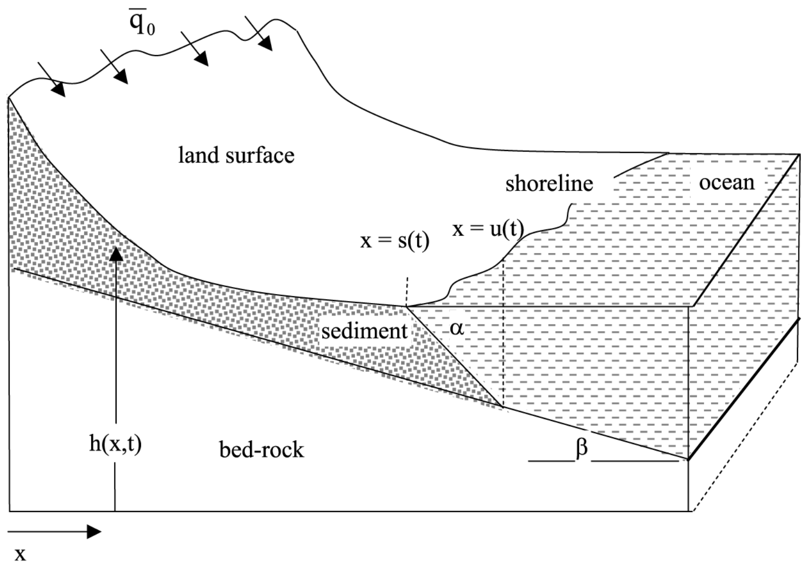

Figure 1.
Schematic of sediment ocean basin

$$\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial x^2}, \quad 0 \leq x \leq s(t) \tag{1}$$

where $s(t)$ is the position of the shoreline, h is the height of the sediment surface above a datum and the diffusivity ν depends on the characteristics of the sediment grains and the time-averaged water line-discharge over the fluvial surface. Appropriate boundary conditions on equation (1) are:

$$\nu \frac{\partial h}{\partial x}\bigg|_{x=0} = -\overline{q}(t), \quad \text{and} \quad h|_{x=s(t)} = 0 \tag{2}$$

where $\overline{q}$ is a prescribed sediment line-flux entering the system. The problem is closed by invoking the sediment balance on the moving shoreline. This requires a treatment of the offshore sub-aqueous sediment transport. In the physically validated model proposed by Swenson *et al.* (2000), it is assumed that the time scale for grain movement by sub-aqueous avalanches is much smaller than for fluvial processes in the sub-aerial domain. This sets the offshore sediment surface at a fixed slope of repose α and the off-shore deposit forms a sediment wedge. A sediment balance at the shoreline then equates the sediment flux arriving at the shoreline to rate at which the submarine sediment wedge can be moved, i.e.

$$\nu \frac{\partial h}{\partial x}\bigg|_{x=s} = -\gamma s \frac{\mathrm{d}s}{\mathrm{d}t} \tag{3}$$

where through geometric construction:

$$\alpha(u - s) = \frac{\alpha\beta}{\alpha - \beta} s = \gamma s \tag{4}$$

is identified as the depth of the ocean at the location of the moving sediment toe, $x = u(t)$ – the location where the submarine sediment wedge intersects the ocean basement, see Figure 1. Equation (3) also follows from the more general shoreline condition, accounting for tectonic subsidence and ocean level change presented by Swenson *et al.* (2000). Also note that, on defining a latent heat term as $L = \gamma s$, (equation (3)) has the form of a one-phase Stefan melting condition (Crank, 1984). For this reason we refer to equation (3) as the shoreline-Stefan condition; the distinguishing feature being the space dependent latent heat term γs.

3. An analytical solution

Voller *et al.* (2004) have derived a similarity solution for the problem defined by equations (1)-(3). Briefly, on rewriting these equations in terms of the similarity variable:

$$\xi = \frac{x}{2t^{1/2}}, \tag{5}$$

and the scaled sediment height:

$$\eta = \frac{h}{2t^{1/2}}, \tag{6}$$

the governing equation for fluvial transport can be written as an ODE. The solution is:

$$\eta(\xi) = \frac{\bar{q}}{\nu}\left[\lambda\left(\frac{e^{-\xi^2/\nu} + \pi^{1/2}\nu^{-1/2}\xi\,\mathrm{erf}(\xi\nu^{-1/2})}{e^{-\lambda^2/\nu} + \pi^{1/2}\nu^{-1/2}\lambda\,\mathrm{erf}(\lambda\nu^{-1/2})}\right) - \xi\right] \tag{7}$$

and, on using the shoreline-Stefan condition (equation (3)), the shoreline movement is given by:

$$s = 2\lambda t^{1/2} \tag{8}$$

where the constant λ is obtained on solution of:

$$\frac{\pi^{1/2}\nu^{-1/2}\mathrm{erf}(\lambda\nu^{-1/2})}{e^{-\lambda^2/\nu} + \pi^{1/2}\nu^{-1/2}\lambda\,\mathrm{erf}(\lambda\nu^{-1/2})} = \frac{1}{\lambda} - \frac{2\gamma\lambda}{\bar{q}} \tag{9}$$

4. A general shoreline problem

The problem outlined in Sections 1 and 2 is restricted to a one-dimensional movement of the shoreline, a specified non-subsiding ocean floor (basement), and a constant ocean level. These restrictions can be removed and the close connection to heat transfer problems retained under the assumption of a steep ocean wedge, $\alpha \to \infty$, see Figure 2. This assumption – justified by the observation that submarine sediment slopes are typically much steeper than fluvial (river) slopes – leads to a governing fluvial transport equation:

$$\frac{\partial h}{\partial t} = \nabla \cdot (\nu \nabla h) - \frac{\partial \Gamma}{\partial t} \tag{10}$$

where the datum is the sea level, the operator:

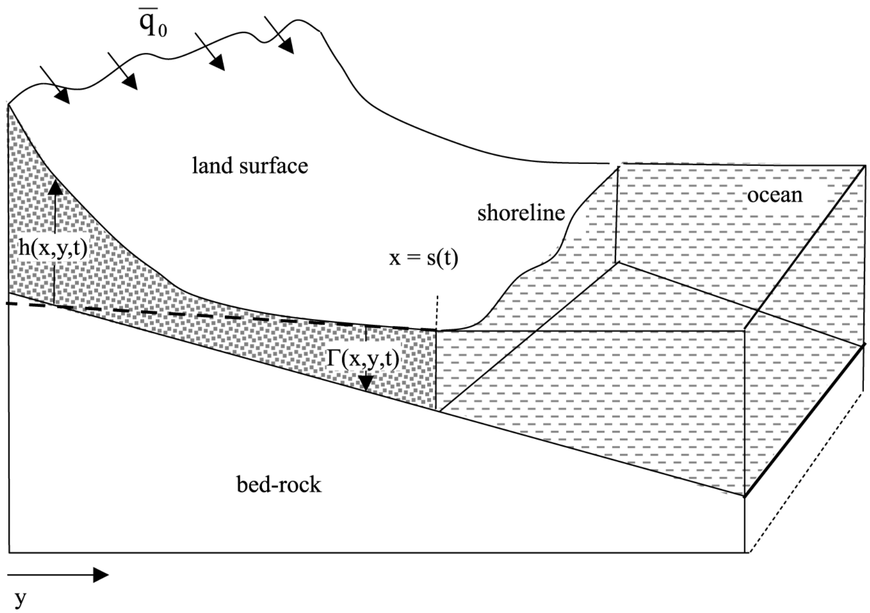

Figure 2.
Ocean basin with cliff-face shoreline

$$\nabla \equiv \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \Gamma(x, y, t)$$

measures the depth of the sediment below sea-level – on the shoreline this value is the ocean depth Γ_s – and temporal changes in Γ account for subsidence or relative sea-level changes. The domain of equation (10) is the sub-aerial fluvial surface. Sediment is supplied to the system by specifying point source on the boundary. On the shoreline boundary of the fluvial domain the shoreline-Stefan balance can be written as:

$$-\nu \nabla h \cdot \mathbf{n} = \Gamma_s \mathbf{v} \cdot \mathbf{n} \tag{11}$$

where **n** is the normal on the shoreline pointing into the ocean and **v** is the velocity of the shoreline. Equations (10) and (11) are closely related to the two-dimensional Stefan problem counterparts; the difference is the subsidence sink term in equation (10) and the space and time dependent latent heat term in equation (11).

5. An enthalpy formulation

The close connection between the shoreline problem of Section 4 and the classic Stefan problem allows for the application of the extensive moving boundary technologies developed for the classic problem. In this case, we are interested in developing fixed grid enthalpy methods. For the shoreline problem of Section 4, an enthalpy function is defined as:

$$H = h + \Gamma \tag{12}$$

To use equation (12) to arrive at an appropriate enthalpy method consider an arbitrary closed two-dimensional control area A, made up of a fluvial A_f and a submarine A_a contribution, separated by the shoreline $s(x, y, t) = 0$, Figure 3. The closed surface of the control area $S = S_f$ (the contiguous section in the fluvial domain) + S_a (the contiguous section in the submarine domain). On noting that in the submarine domain:

$$\int_{A_a} H \mathrm{d}A = 0 \tag{13}$$

and on its surface S_a,$\nabla h \cdot \mathbf{n} = 0$, a sediment balance on the control area can be written as:

$$\frac{\partial}{\partial t}\int_{A_f} H \mathrm{d}A = \int_{S_f} \nu \nabla h \cdot \mathbf{n}\, \mathrm{d}S \tag{14}$$

where $\mathbf{n}$ is the outward pointing normal on S. On noting that $h = 0$ on the shoreline boundary, and using the Leibniz rule (Reynolds transport theorem) on the left of equation (14):

$$\int_{A_f} \frac{\partial H}{\partial t}\, \mathrm{d}A + \int_{s} \Gamma_s \mathbf{v} \cdot \mathbf{n}\, \mathrm{d}S = \oint_{S_f + s} \nu \nabla h \cdot \mathbf{n}\, \mathrm{d}S - \int_{s} \nu \nabla h \cdot \mathbf{n}\, \mathrm{d}S \tag{15}$$

By the shoreline-Stefan condition in equation (11) the integrals over the shoreline section of the boundary cancel out, and since the boundary $S_f + s$ encloses the area A_f, the divergence theorem can be used to arrive at:

$$\int_{A_f} \frac{\partial H}{\partial t} - \nabla \cdot (\nu \nabla h) \mathrm{d}A = 0 \tag{16}$$

Since, the chosen control area is arbitrary the argument of equation (16) has to be zero everywhere, i.e.

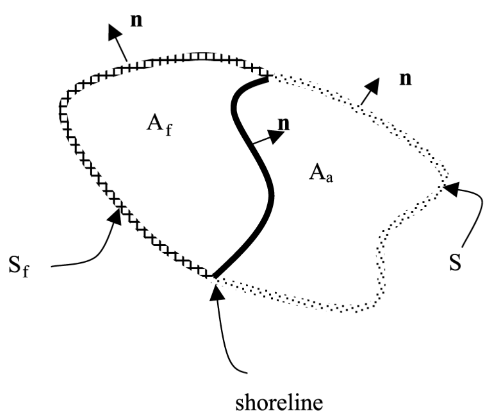

Figure 3.
Arbitrary control area

$$\frac{\partial H}{\partial t} = \nabla \cdot (\nu \nabla h) \tag{17}$$

with, by equation (12):

$$h = \begin{cases} H - \Gamma & \text{if } H \geq \Gamma \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

The formulation in equations (17) and (18), which is applicable thought out the entire domain (fluvial and submarine) matches the basic enthalpy formulation, e.g. see Crank (1984), it is critical to note, however, that in this case the latent heat term Γ can be a function of space (a variable ocean basement) and time (subsidence or ocean level change).

6. A fixed grid solution

Numerical solutions of equations (17) and (18) can developed from conventional enthalpy methods, e.g. time explicit, apparent specific heat, and source based (Crank, 1984; Voller, 1996). These methods can be based on both finite element and finite difference discretizations. The basic time explicit approach can be applied without modification. Owing to the space and time variations of the latent heat term, however, implicit time integration schemes based on an apparent specific heat or source term could require significant modification.

As an example, a basic explicit time integration finite difference solution of equations (17) and (18) is developed. The solution domain is considered to be a two-dimensional rectangular region in the *x*-*y* plane of the ocean level. As shown in Figure 4, this region is covered by a grid of conforming square control volumes of side length Δ; a node point is placed in the center of each volume. At the node point on row i and column j, removed from the domain boundaries, an explicit time integration of equation (17) is:

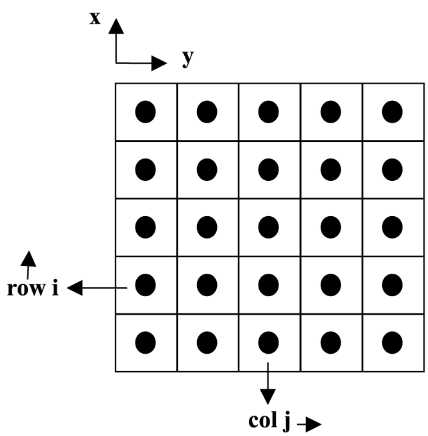

Figure 4.
A representative grid of square control volunes

$$H_{ij}^{\text{new}} = H_{ij} + \frac{\Delta t}{\Delta^2}\left[\nu_w(h_{ij-1} - h_{ij}) - \nu_e(h_{ij} - h_{ij+1}) + \nu_s(h_{i-1j} - h_{ij}) - \nu_n(h_{ij} - h_{i+1j})\right] \tag{19}$$

where ν_w indicates evaluation of the diffusivity at the interface between nodes $(i, j-1)$ and (i, j), etc. and Δt, the simulation time step, is chosen such that the largest value of $\nu\Delta t \leq 0.25\Delta^2$ to ensure a stable scheme. At nodes in control volumes located along the domain edges the appropriate terms on the right hand side of equation (19) are dropped, e.g. along the west domain edge $(j = 1)$

$$\nu_w(h_{ij-1} - h_{ij}) = 0 \tag{20}$$

The scheme equation (19) also needs to be modified at nodes where a sediment source is applied. For example, if a source q m^3/s is applied at the southwest domain corner the scheme at node $i = 1, j = 1$ is written as:

$$H_{11}^{\text{new}} = H_{11} + \frac{\Delta t}{\Delta^2}\left[q - \nu_e(h_{11} - h_{12}) - \nu_n(h_{11} - h_{21})\right] \tag{21}$$

In a time step, the current nodal values of the sediment depth below sea level Γ_{ij} – values that follow a prescribed path – are calculated. Then equation (19) and its variants are solved to provide an update of the sediment enthalpy at each domain node point. Following, the nodal sediment heights above sea-level are calculated from a discretization of equation (18), i.e.

$$h_{ij}^{\text{new}} = \begin{cases} H_{ij}^{\text{new}} - \Gamma_{ij} & \text{if } H_{ij}^{\text{new}} > \Gamma_{ij} \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

which completes the time step calculations.

7. Results

An initial test problem of the above scheme is for the case where a line source $\overline{q} = 1\,\text{m}^3/\text{m}$ is applied on the south edge $(i = 1, y = 0)$ of the domain and the ocean slopes away from this edge with a constant slope β In this case, the sediment depth below sea level is a function of y alone, i.e. $\Gamma(x, y, t) = y\beta$. Further, the shoreline movement is one-dimensional and can be analytically determined by substituting $\gamma = \beta$ in equations (8) and (9). Numerical predictions of the shoreline movement are obtained using a grid of 100×100 square control volumes of unit size and a time step of 0.1; grid dependence studies have shown these to be sufficient choices. A numerical position of the shoreline is determined by locating the control volume row where the shoreline is located, i.e. the row where $0 < H_i < (i - 0.5)\Delta\beta$ and setting:

$$s = (i - 1)\Delta + \frac{H_i}{\Gamma_i}\Delta \tag{23}$$

Numerical predictions for the shoreline movement, for various values of β are shown in Figure 5, the position is recorded when the shoreline crosses a line of nodes. The symbols in this figure are the numerical predictions the lines the analytical solution. The accuracy of these results clearly verifies the enthalpy scheme in equations (19)-(22).

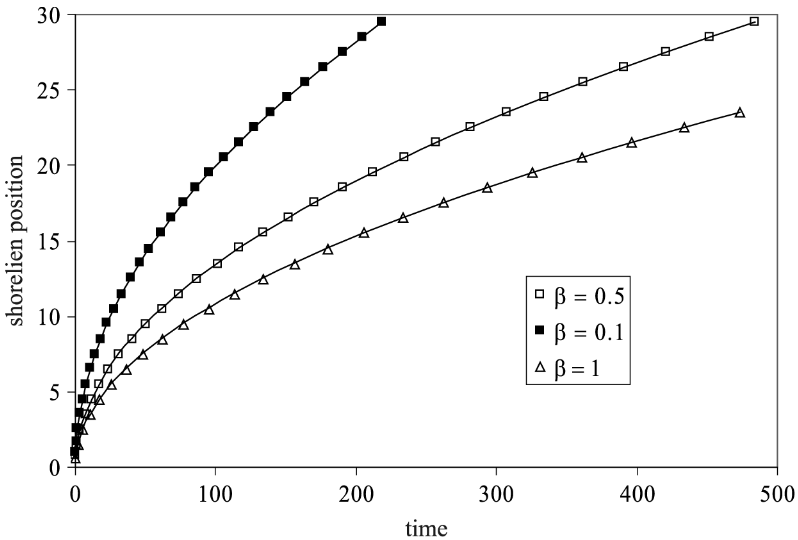

Figure 5.
Shoreline positions with time for a shoreline growing into a constant sloping ocean (slope β). The symbols are numerical predictions the lines analytical values

An immediate extension of the previous problem is to introduce a temporal change in addition to a spatial change by setting:

$$\Gamma(x, y, t) = y\beta(1 + \theta t) \tag{24}$$

This represents an ocean in which the basement hinges about the line $y = 0$, such that the value of the y-slope increases with time. There is no analytical solution in this case. It is noted, however, that as time advances, the rate of accommodation space, created under the fluvial deposit by basement subsidence, approaches the rate of sediment supply across $y = 0$. Balancing these rates, assuming a constant values of θ,

$$\bar{q} = \frac{s^{\mathrm{up}^2}\beta\theta}{2} \tag{25}$$

gives on rearrangement an upper bound on the shoreline position:

$$s^{\mathrm{up}} = \sqrt{\frac{2\bar{q}}{\beta\theta}} \tag{26}$$

At initial stages of a simulation, the shoreline position will follow the $\sqrt{t}$ dependence. As time increases, however, the shoreline position will asymptotically approach the fixed value in equation (26). This behavior is clearly seen in Figure 6 which plots the numerical predictions of the shoreline movement under the conditions $\beta = 0.1$ and $\theta = 0.025$.

The one-dimensional shoreline movement results, presented above, clearly demonstrate that the proposed enthalpy shoreline tracking scheme is able to predict the correct limit case behaviors. Encouraged with this performance, a more general two-dimensional shoreline problem can be constructed by applying sediment source terms of strength $q = 1$ at the center nodes $i = 1, j = 50$ and $i = 1, j = 51$. This setting

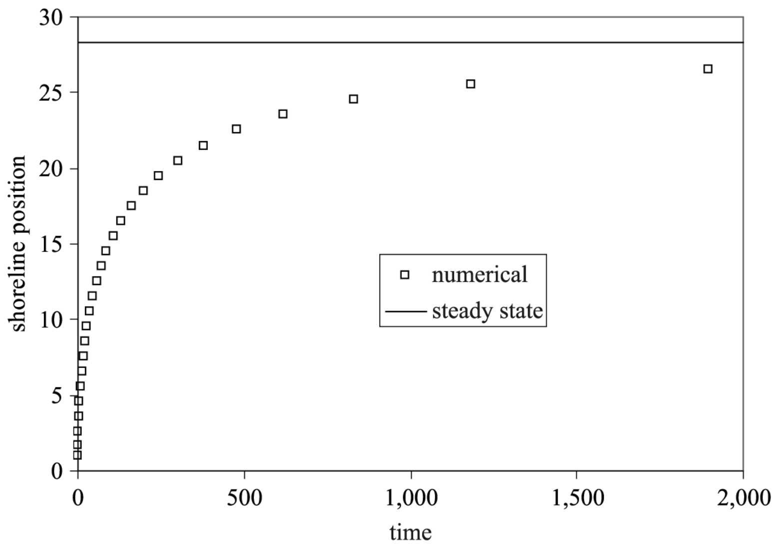

Figure 6. Shoreline position with time for a shoreline growing into a ocean undergoing hinge subsidence

will lead to two-dimensional ocean front $s(x, y, t)$, whose exact shape is controlled by the choice of behavior for the ocean basement. Two ocean basement models are considered. The first sets a constant fixed depth ocean:

$$\Gamma(x, y, t) = 0.175; \tag{27}$$

a problem that exactly matches a heat transfer melting problem. The second imposes an ocean basin that varies in space and time according to:

$$\Gamma(x, y, t) = 0.175 + \begin{cases} yt/1,000, & 44.5 < x < 55.5 \\ 0, & \text{otherwise} \end{cases} \tag{28}$$

The ocean basement in this problem is initially at uniform depth, but as time advances a centerline trench is formed by a hinged subsidence (equation (24)). The formation of this trench, corresponding to a locally increasing latent heat value, should retard the movement of the shoreline (melt front) along the center line of the domain.

Figure 7 shows predictions for the shore line position at simulation times of 100, 200, 400, 800, and 1,600 for the constant basement ocean, Γ given by equation (27). The behavior is as expected; matching what would be observed for melting around a constant heat source. In contrast, Figure 8 shows the shoreline positions, at identical times, when the basement Γ accounts for the developing trench, equation (28). In this case, a notch, “cove” like feature forms where the shoreline advance is retarded by the trench development. Figure 9 shows a three-dimensional image of the above sea level surface at simulation time $t = 1,600$ for this last case.

8. Conclusions

The solution of moving boundary problems related to melting has long been a major research theme in numerical heat transfer applications. Recently (Swenson *et al.*, 2000)

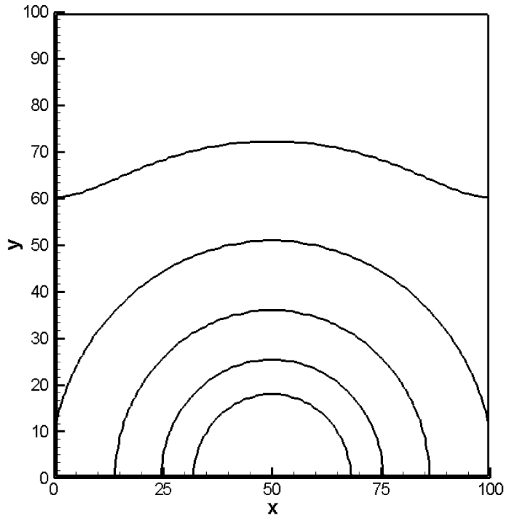

Figure 7.
Shoreline positions at time $t = 100$, 200, 400, 800, and 1,600 advancing from as single input point into a constant depth ocean

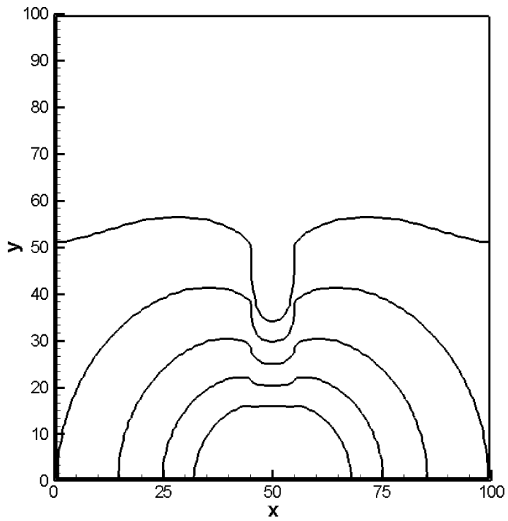

Figure 8.
Shoreline positions at time $t = 100$, 200, 400, 800, and 1,600 advancing from as single input point into an ocean with a centerline trench evolving by hinge subsidence

it was recognized that melting problems are analogous to the geological time scale movement of the shoreline in sedimentary basins. This paper has show that particular versions of the shoreline model can be posed as a melting problem involving a space and time dependent latent heat. An enthalpy model has been developed and an associated fixed-grid, time-explicit algorithm has been verified by comparison with an analytic similarity solution and a limit case geometric solution. This work has only required a modest amount of method development, the novelty in the work firmly rests in solving physically relevant, melting-like problems, which, through the specification of a space and time dependent latent heat, contain unique features in the resulting moving boundaries.

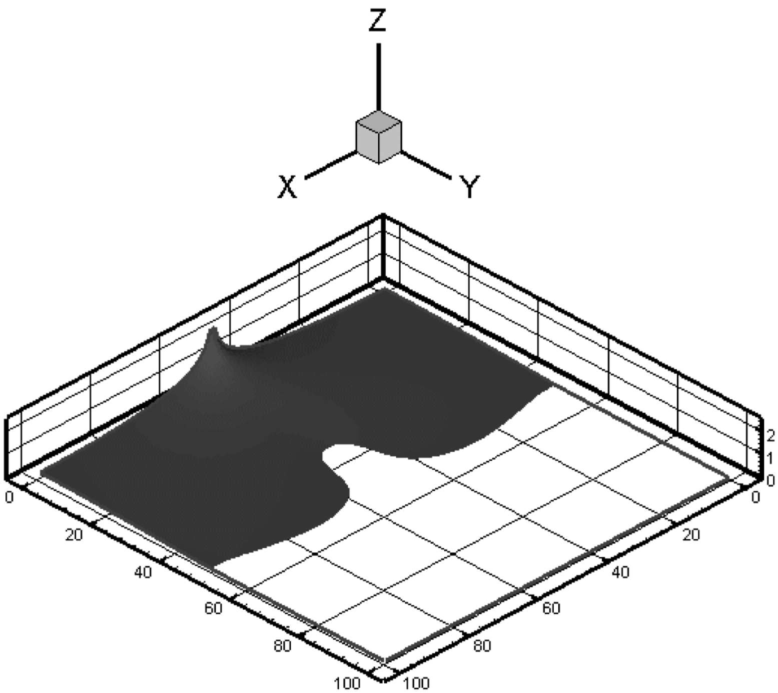

Figure 9. Sediment above sea-level at time $t = 1,600$ for the shoreline problem given in Figure 8

Further work will include:

- Incorporation of a finite sloping sub-aqueous wedge into the fixed grid scheme. This is an interesting case to investigate because such a feature can lead to so-called "auto-retreat" (Swenson *et al.*, 2000) of the shoreline – a point is reached where the sediment flux can no longer sustain the shoreline position resulting a retreat of the shoreline towards the shore.
- The development of implicit time solvers. A step, due to the variable latent heat, that may require significant method development.

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